

Grassmann Manifold $G(2, 8)$ and Complex Structure on S^6

Jianwei Zhou

Department of Mathematics, Suzhou University, Suzhou 215006, P.R. China

Abstract

In this paper, we use Clifford algebra and the spinor calculus to study the complex structures on Euclidean space R^8 and the spheres S^4, S^6 . By the spin representation of $G(2, 8) \subset Spin(8)$ we show that the Grassmann manifold $G(2, 8)$ can be looked as the set of orthogonal complex structures on R^8 . In this way, we show that $G(2, 8)$ and CP^3 can be looked as twistor spaces of S^6 and S^4 respectively. Then we show that there is no almost complex structure on sphere S^4 and there is no orthogonal complex structure on the sphere S^6 .

§1. Introduction

In this paper, we study the complex structures on Euclidean space R^8 and the spheres S^4, S^6 respectively. In the study we use Clifford algebra and the spinor calculus. The main references are [4], [5], [6] and [7].

Let $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_8$ be a fixed orthonormal basis of R^8 , the Clifford product on Clifford algebra $C\ell_8$ be determined by the relations:

$$\bar{e}_B \bar{e}_C + \bar{e}_C \bar{e}_B = -2\delta_{BC}, \quad B, C = 1, 2, \dots, 8.$$

As in [6], let $A_8 = Re[(\bar{e}_1 + \sqrt{-1}\bar{e}_2) \cdots (\bar{e}_7 + \sqrt{-1}\bar{e}_8)]$, $\beta_8 = \bar{e}_1 \bar{e}_3 \bar{e}_5 \bar{e}_7$, $A = A_8(1 + \beta_8)$. The space $V = C\ell_8 \cdot A = V^+ \oplus V^-$ is an irreducible module over $C\ell_8$ and the spinor spaces $V^+ = C\ell_8^{even} A$, $V^- = C\ell_8^{odd} A$ are generated by $\alpha_B = \bar{e}_1 \bar{e}_B A$ and $\alpha_{B+8} = \bar{e}_B A$ respectively, $B = 1, \dots, 8$. For any $x \in C\ell_8$,

$$x(\alpha_1, \dots, \alpha_8, \alpha_9, \dots, \alpha_{16}) = (\alpha_1, \dots, \alpha_8, \alpha_9, \dots, \alpha_{16})\Phi(x)$$

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defines an algebra isomorphism $\Phi: Cl_8 \rightarrow R(16)$.

As shown in [6], for $x \in Cl_8$, we have $\Phi(\alpha(x^t)) = (\Phi(x))^t$. Then for any $g \in Spin(8)$, $\Phi(g) = \begin{pmatrix} B & \\ & C \end{pmatrix}$, we have $B, C \in SO(8)$. If $g^2 = -1$, B and C define orthogonal almost complex structures on R^8 .

In this paper, we use isomorphism $\Phi: Cl_8 \rightarrow R(16)$ to construct homeomorphism Φ^* from Grassmann manifold $G(2, 8)$ to the set of oriented orthogonal complex structures on R^8 . Furthermore, there are two fibre bundles $\tau: G(2, 8) \rightarrow S^6$ and $\tau_1: CP^3 \rightarrow S^4$ defined naturally. We show that restricting the homeomorphism Φ^* on the fibres of these fibre bundles respectively, we get the sets of complex structures on the tangent spaces of S^6 and S^4 respectively. By definition on [4] p.339, $G(2, 8)$ and CP^3 are the twistor spaces on S^6 and S^4 respectively.

Then the almost complex structure on the sphere S^6 and S^4 are determined by sections $f: S^6 \rightarrow G(2, 8)$ and $f: S^4 \rightarrow CP^3$ respectively. By the cohomology groups of CP^3 and S^4 , we show that there is no almost complex on S^4 .

$G(2, 8)$ is a Kaehler manifold, in §3 we show that the almost complex structure on S^6 is integrable if and only if the map $f: S^6 \rightarrow G(2, 8)$ is holomorphic. Then S^6 is also a Kaehler manifold if $f: S^6 \rightarrow G(2, 8)$ is holomorphic. These show there is no orthogonal complex structure on the sphere S^6 .

For the complex structures on R^8, S^6, S^4 see also [1] and [3], p.159, p. 281.

§2. The complex structures on R^8

$G(2, 8)$ is the Grassmann manifold formed by all oriented 2-dimensional subspaces of R^8 , any $x \in G(2, 8)$ can be represented by $e_1 e_2$ or $e_1 \wedge e_2$, where e_1, e_2 is an oriented orthonormal basis of x . Then $G(2, 8)$ can be viewed as a subspace of $Spin(8)$ or $\Lambda^2(R^8)$. Let

$$M = \{A \in SO(8) \mid A^2 = -I\} \approx SO(8)/U(4)$$

be the set of oriented orthogonal complex structures on R^8 .

Theorem 2.1 The map $\Phi^*: G(2, 8) \rightarrow M$ defined by

$$x(\alpha_1, \dots, \alpha_8) = (\alpha_1, \dots, \alpha_8)\Phi^*(x), \quad x \in G(2, 8),$$

is a diffeomorphism. $\Phi^*(x)$ is the spin representation of x .

Proof For any $x \in G(2, 8)$, $\Phi(x) = \begin{pmatrix} B & \\ & C \end{pmatrix}$, where $B, C \in SO(8)$. From $x \cdot x = -1$, we have $BB = -I$, B defines a complex structure on $R^8 \cong V^+$.

By Proposition 4.3 of [7], the map $x \in G(2, 8) \mapsto B$ is a monomorphism. The proposition follows from

$$G(2, 8) = \{A(\bar{e}_1)A(\bar{e}_2) \mid A \in SO(8)\} = \{g(\bar{e}_1\bar{e}_2)g^t \mid g \in Spin(8)\},$$

$$\Phi(g(\bar{e}_1\bar{e}_2)g^t) = \Phi(g)\Phi(\bar{e}_1\bar{e}_2)\Phi(g)^t,$$

and

$$M = \{ABA^t \mid A \in SO(8)\},$$

where $B \in M$ is a fixed element.

For any $v \in R^8$ there is $P_v \in SO(8)$ such that $\Phi(v) = \begin{pmatrix} & P_v \\ -P_v^t & \end{pmatrix}$. Then for any $x = e_1e_2 \in G(2, 8)$, $\Phi(x) = -P_{e_1}P_{e_2}^t$. Furthermore, $P_v \in M$ if $v \in S^7, v \perp \bar{e}_1$.

Denote $J_x: R^8 \rightarrow R^8$ the complex structure defined by $x \in G(2, 8)$. For any $v = \sum_{i=1}^8 v^i \bar{e}_i \in R^8$, $\bar{e}_1 v A = \sum_{i=1}^8 v^i \alpha_i$, we have

$$\bar{e}_1 J_x v A = x \bar{e}_1 v A.$$

It is easy to compute

$$\Phi^*(\bar{e}_1\bar{e}_2) = P_{\bar{e}_2} = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{pmatrix}.$$

The complex structure $J_x = \Phi^*(x)$ acts on the left of R^8 .

As shown in [5] or [7], for any $x \in G(2, 8)$ there is $v \in S^6 = \{v \in S^7 \mid v \perp \bar{e}_1\}$ such that $xA = \bar{e}_1 v A$, $x \mapsto v$ defines a fibre bundle $\tau: G(2, 8) \rightarrow S^6$. For any $v \in S^6$, we have $\tau^{-1}(v) = \{u J_v u \mid u \in S^7\}$, where J_v is a complex structure on R^8 , see [7] or §3 below. The fibres of τ are diffeomorphic to the complex projective space CP^3 .

Theorem 2.2 For any $v \in S^6$, $\Phi^*(\tau^{-1}(v))$ is the set of orthogonal almost complex structures on the tangent space $T_v S^6$. Then $G(2, 8)$ is the twistor space of S^6 and any almost complex structure on S^6 is defined by a section $f: S^6 \rightarrow G(2, 8)$.

Proof For any $v \in S^6$, $T_v S^6 = \{X \in R^8 \mid X \perp \bar{e}_1, v\}$. On the other hand, $x \in \tau^{-1}(v)$ if and only if

$$xA = -x\alpha_1 = \bar{e}_1 v A, \quad x\bar{e}_1 v A = xx A = -A.$$

Then the subspace of R^8 generated by \bar{e}_1, v is invariant under the map $J_x = \Phi^*(x)$ for any $x \in \tau^{-1}(v)$ and J_x gives a complex structure on $T_v S^6$. By Theorem 2.1, all complex structures on the tangent space $T_v S^6$ can be obtained in this way.

By this theorem, any $x \in \tau^{-1}(v)$ gives a complex structure $J_x: T_v S^6 \rightarrow T_v S^6$. For any $X \in T_v S^6$, $Y = J_x X$ is defined by $x\bar{e}_1 X A = \bar{e}_1 Y A$.

By definition on [4] p.339, $G(2, 8)$ is a twistor space of S^6 , any almost complex structure on S^6 is defined by a section $f: S^6 \rightarrow G(2, 8)$. In §3, we shall make further study and show that there is no integrable complex structure on S^6 .

As is well-known, there is a map $CP^3 \rightarrow S^4$. This map can also be constructed by the Clifford algebra $C\ell_8$. In the following we consider $\tau^{-1}(\bar{e}_3) \approx CP^3$ as an example.

Let J be a complex structure on R^8 defined by $J\bar{e}_{2i-1} = \bar{e}_{2i}$, $J\bar{e}_{2i} = -\bar{e}_{2i-1}$, $i = 1, 2, 3, 4$. By Lemma 3.6, Proposition 3.7 of [7], for any $x \in \pi^{-1}(\bar{e}_3)$, there is a vector v such that

$$xA_8\beta_8 = \bar{e}_1 v A_8 + \bar{e}_1(\bar{e}_3 - v)A_8\beta_8,$$

and $v \perp \bar{e}_1, \bar{e}_2, \bar{e}_4$, $|\bar{e}_3 - 2v| = 1$. From this equation we have

$$xA_8(1 + \beta_8) = \bar{e}_1 \bar{e}_3 A_8(1 + \beta_8),$$

$$xA_8(1 - \beta_8) = \bar{e}_1(\bar{e}_3 - 2v)A_8(1 - \beta_8).$$

Let $S^4 = \{t = \bar{e}_4 - 2Jv \in S^7 \mid v \perp \bar{e}_1, \bar{e}_2, \bar{e}_4\}$ be a unit sphere in R^8 . The maps $x \mapsto \bar{e}_3 - 2v \mapsto \bar{e}_4 - 2Jv$ define a map

$$\tau_1: \tau^{-1}(\bar{e}_3) \approx CP^3 \rightarrow S^4.$$

By Proposition 2.6 of [7], the 2-form of $xA_8\beta_8 = \bar{e}_1 v A_8 + \bar{e}_1(\bar{e}_3 - v)A_8\beta_8$ is a calibration and its contact set diffeomorphic to CP^1 which is also the set $\tau_1^{-1}(t)$. Then $\tau_1: \tau^{-1}(\bar{e}_3) \rightarrow S^4$ is a fibre bundle.

Theorem 2.3 Restricting the map $\Phi^*: G(2, 8) \rightarrow M$ on the fibre $\tau_1^{-1}(t)$ of τ_1 , we have all complex structures on the tangent space $T_t S^4$. Then CP^3 is the twistor space of S^4 .

Proof It is easy to see $A_8(1 - \beta_8)\bar{e}_1\bar{e}_2 = -\bar{e}_1\bar{e}_2 A_8(1 + \beta_8) = -\alpha_2$, from $xA_8(1 - \beta_8) = \bar{e}_1(\bar{e}_3 - 2v)A_8(1 - \beta_8)$ and $\bar{e}_1\bar{e}_2 A_8(1 + \beta_8) = vJv A_8(1 + \beta_8)$ for any $v \in S^7$, we have

$$\begin{aligned} x\alpha_2 &= \bar{e}_1(\bar{e}_3 - 2v)\bar{e}_1\bar{e}_2 A_8(1 + \beta_8) \\ &= \bar{e}_1(\bar{e}_3 - 2v)\bar{e}_3\bar{e}_4 A_8(1 + \beta_8) \\ &= -\bar{e}_1\bar{e}_4 A_8(1 + \beta_8) - 2\bar{e}_1 v v Jv A_8(1 + \beta_8) \\ &= -\bar{e}_1(\bar{e}_4 - 2Jv)A_8(1 + \beta_8). \end{aligned}$$

Thus for any $x \in G(2, 8)$, $xA_8\beta_8 = \bar{e}_1vA_8 + \bar{e}_1(\bar{e}_3 - v)A_8\beta_8$ if and only if the following identities hold.

$$x\alpha_1 = -xA_8(1 + \beta_8) = -\alpha_3,$$

$$x\alpha_3 = \alpha_1,$$

$$x\alpha_2 = -\bar{e}_1(\bar{e}_4 - 2Jv)A_8(1 + \beta_8),$$

$$x\bar{e}_1(\bar{e}_4 - 2Jv)A_8(1 + \beta_8) = \alpha_2.$$

These shows that the subspace of R^8 generated by $\bar{e}_1, \bar{e}_2, \bar{e}_3, t = \bar{e}_4 - 2Jv$ is invariant under the action defined by J_x . Thus J_x defines a complex structure on $T_tS^4 = \{ w \in R^8 \mid w \perp \bar{e}_1, \bar{e}_2, \bar{e}_3, t \}$.

Then any almost complex structure on S^4 defines a section of the fibre bundle $\tau_1: \tau^{-1}(\bar{e}_3) \rightarrow S^4$. We have proved Theorem.

In the following we study the fibres of $\tau_1: \tau^{-1}(\bar{e}_3) \approx CP^3 \rightarrow S^4$.

First assuming $t = \cos \theta \bar{e}_4 + \sin \theta \bar{e}_6 \in S^4$. From $\bar{e}_3 - 2v = \cos \theta \bar{e}_3 + \sin \theta \bar{e}_5$, we have

$$v = \sin \frac{\theta}{2} (\sin \frac{\theta}{2} e_3 - \cos \frac{\theta}{2} e_5), \quad e_3 - v = \cos \frac{\theta}{2} (\sin \frac{\theta}{2} \bar{e}_3 + \sin \frac{\theta}{2} \bar{e}_5).$$

Then

$$\begin{aligned} & \bar{e}_1vA_8 + \bar{e}_1(\bar{e}_3 - v)A_8\beta_8 \\ &= \bar{e}_1 \sin \frac{\theta}{2} (\sin \frac{\theta}{2} e_3 - \cos \frac{\theta}{2} e_5) \bar{e}_1 \bar{e}_3 \bar{e}_5 \bar{e}_7 A_8 \beta_8 \\ & \quad + \bar{e}_1 \cos \frac{\theta}{2} (\sin \frac{\theta}{2} \bar{e}_3 + \sin \frac{\theta}{2} \bar{e}_5) A_8 \beta_8 \\ &= (\cos \frac{\theta}{2} \bar{e}_1 + \sin \frac{\theta}{2} \bar{e}_7) (\cos \frac{\theta}{2} \bar{e}_3 + \sin \frac{\theta}{2} \bar{e}_5) A_8 \beta_8. \end{aligned}$$

By Proposition 2.6 of [7], the calibration defined by $\bar{e}_1vA_8 + \bar{e}_1(\bar{e}_3 - v)A_8\beta_8$ is

$$(\cos \frac{\theta}{2} \bar{e}_1 + \sin \frac{\theta}{2} \bar{e}_7) (\cos \frac{\theta}{2} \bar{e}_3 + \sin \frac{\theta}{2} \bar{e}_5) - (\cos \frac{\theta}{2} \bar{e}_2 + \sin \frac{\theta}{2} \bar{e}_8) (\cos \frac{\theta}{2} \bar{e}_4 + \sin \frac{\theta}{2} \bar{e}_6).$$

This is a special Lagrangian calibration and the set $\tau_1^{-1}(t)$ is the contact set of this calibration which diffeomorphic to CP^1 . For any $x \in \tau_1^{-1}(t)$, we have

$$\begin{aligned} xA_8\beta_8 &= \bar{e}_1vA_8 + \bar{e}_1(\bar{e}_3 - v)A_8\beta_8 \\ &= (\cos \frac{\theta}{2} \bar{e}_1 + \sin \frac{\theta}{2} \bar{e}_7) (\cos \frac{\theta}{2} \bar{e}_3 + \sin \frac{\theta}{2} \bar{e}_5) A_8 \beta_8. \end{aligned}$$

Let V be a subspace of R^8 generated by

$$\cos \frac{\theta}{2} \bar{e}_1 + \sin \frac{\theta}{2} \bar{e}_7, \quad \cos \frac{\theta}{2} \bar{e}_3 + \sin \frac{\theta}{2} \bar{e}_5, \quad \cos \frac{\theta}{2} \bar{e}_2 + \sin \frac{\theta}{2} \bar{e}_8, \quad \cos \frac{\theta}{2} \bar{e}_4 + \sin \frac{\theta}{2} \bar{e}_6.$$

Then any element of $\tau_1^{-1}(t)$ can be represented by $vJ_{\bar{e}_3}v$, $v \in V$.

For general $u \in S^4$, we can choose an element $G \in U(2)$ such that $G(u) = t = \cos \theta \bar{e}_4 + \sin \theta \bar{e}_6$, where $U(2) \subset Spin_7$ is the unitary group which fixed the elements $\bar{e}_1, \dots, \bar{e}_4$. Since $A_8(1 + \beta_8)$ is invariant under the action by $Spin_7$, we have

$$\tau_1^{-1}(u) = G^{-1}(\tau_1^{-1}(t)).$$

Corollary 2.4 There is no almost complex structure on the sphere S^4 .

Proof If there is an almost complex structure on the sphere S^4 , we have a section $f: S^4 \rightarrow CP^3$. As we know the cohomology $H^*(CP^3, Z)$ is generated by an element $a \in H^2(CP^3, Z)$, $f^*a = 0$, then $f^*(a \cup a) = 0$. Let $[\xi] \in H^4(S^4)$ be a generator, $\tau_1^*[\xi] = \lambda a \cup a$. We have $f^*\tau_1^*[\xi] = \lambda f^*(a \cup a) = 0$, this contradicts to the fact $\tau_1 \circ f = id$. These shows there is no almost complex structure on the sphere S^4 .

For Corollary 2.4, see also [2].

By Theorem 2.1, 2.2, 2.3 and [4] p.342, we have

$$G(2, 8) = SO(8)/SO(2) \times SO(6) \approx SO(8)/U(4) \approx SO(7)/U(3),$$

$$CP^3 = U(4)/U(1) \times U(3) \approx SO(6)/U(3) \approx SO(5)/U(2),$$

$$CP^1 = U(2)/U(1) \times U(1) \approx SO(4)/U(2).$$

§3. The complex structures on S^6

In the following we show that there is no complex structure on the sphere S^6 . First we study the differential geometry on the fibre bundle $\tau: g(2, 8) \rightarrow S^6$.

Let e_1, e_2, \dots, e_8 be orthonormal frame fields on R^8 such that $e_1 \wedge e_2$ generate a neighborhood of x in $G(2, 8)$. By

$$d(e_1 \wedge e_2) = \sum_{\alpha=3}^8 \omega_1^\alpha E_{1\alpha} + \sum_{\alpha=3}^8 \omega_2^\alpha E_{2\alpha}, \quad \omega_i^\alpha = \langle de_i, e_\alpha \rangle,$$

we know that the elements $E_{1\alpha} = e_\alpha e_2$, $E_{2\alpha} = e_1 e_\alpha$, $\alpha = 3, \dots, 8$, can be looked as a basis of $T_{e_1 e_2} G(2, 8)$, ω_i^α be their dual 1-form. Define the metric on $G(2, 8)$ by

$$ds^2 = 2 \sum_{i=1}^2 \sum_{\alpha=3}^8 (\omega_i^\alpha)^2.$$

Differential $E_{i\alpha}$ we get the Riemannian connection ∇^* on $G(2, 8)$,

$$\nabla^* E_{i\alpha} = \sum_{j=1}^2 \omega_i^j E_{j\alpha} + \sum_{\beta=3}^8 \omega_\alpha^\beta E_{i\beta}.$$

Acting $e_1 e_2$ on $T_{e_1 e_2} G(2, 8)$ by Clifford product defines an almost complex

$$\tilde{J}: TG(2, 8) \rightarrow TG(2, 8),$$

$$\tilde{J}(E_{1\alpha}) = e_1 e_2 e_\alpha e_2 = E_{2\alpha}, \quad \tilde{J}(E_{2\alpha}) = e_1 e_2 e_1 e_\alpha = -E_{1\alpha}.$$

It is easy to see that $(\nabla^* \tilde{J})E_{i\alpha} = \nabla^*(\tilde{J}E_{i\alpha}) - \tilde{J}\nabla^* E_{i\alpha} = 0$. This shows

Proposition 3.1 The almost complex structure \tilde{J} is integrable and makes $G(2, 8)$ a Kaehler manifold.

As is well-known, the Grassmann manifold $G(2, n)$ can be looked as a complex submanifold of CP^{n-1} .

Any $v \in S^6$ defines a complex structure J_v on R^8 , $J_v(\bar{e}_1) = v$, $J_v(v) = -\bar{e}_1$, for any $w \perp \bar{e}_1, v$, $J_v(w)$ is defined by $J_v(w)A = -\bar{e}_1 v w A$ or equivalently $\bar{e}_1 v \bar{e}_1 w A = \bar{e}_1 J_v(w)A$. For any $v \in S^6$, $\tau^{-1}(v) = \{u J_v u \mid u \in S^7\}$.

Proposition 3.2 The map $\tau: G(2, 8) \rightarrow S^6$ is a Riemannian submersion.

Proof It is easy to see that the vertical subspace $T_{e_1 e_2} \tau^{-1}(v)$ of the fibre bundle τ is generated by

$$e_\alpha J_v e_1 + e_1 J_v e_\alpha, \quad \alpha = 3, \dots, 8,$$

where $e_1, e_2 = J_v e_1, e_3, \dots, e_8$ be orthonormal frame fields along $\tau^{-1}(v)$. The orthogonal subspace of $T_{e_1 e_2} \tau^{-1}(v)$ in $T_{e_1 e_2} G(2, 8)$ is generated by

$$e_\alpha J_v e_1 - e_1 J_v e_\alpha, \quad \alpha = 3, \dots, 8.$$

Denote this space by $T_{e_1 e_2}^H G(2, 8)$. By

$$(e_\alpha J_v e_1 + e_1 J_v e_\alpha)A = 0,$$

we have

$$\frac{1}{2}(e_\alpha J_v e_1 - e_1 J_v e_\alpha)A = e_\alpha J_v e_1 A = \bar{e}_1 \tau_*(e_\alpha J_v e_1)A.$$

The norm of $\frac{1}{2}\tau_*(e_\alpha J_v e_1 - e_1 J_v e_\alpha) = \tau_*(e_\alpha J_v e_1) \in T_v S^6$ is 1. For $\alpha \neq \beta$, $\tau_*(e_\alpha J_v e_1) \perp \tau_*(e_\beta J_v e_1)$, we have proved that the map $\tau: G(2, 8) \rightarrow S^6$ is a Riemannian submersion.

Let $TG(2, 8) = T^H G(2, 8) \oplus T^V G(2, 8)$ be the decomposition of tangent space of $G(2, 8)$, $T^H G(2, 8), T^V G(2, 8)$ be horizontal and vertical spaces respectively. The map $\tau_*: T_{e_1 e_2}^H G(2, 8) \rightarrow T_{\tau(e_1 e_2)} S^6$ is an isometric.

Proposition 3.3 (1) The following diagram of maps is commutative,

$$\begin{array}{ccc} T_{e_1 e_2} G(2, 8) & \xrightarrow{\tilde{J}} & T_{e_1 e_2} G(2, 8) \\ \tau_* \downarrow & & \downarrow \tau_* \\ T_v S^6 & \xrightarrow{J_{e_1 e_2}} & T_v S^6; \end{array}$$

(2) $\tilde{J}: T^H G(2, 8) \rightarrow T^H G(2, 8)$, $\tilde{J}: T^V G(2, 8) \rightarrow T^V G(2, 8)$.

Proof For any $\tilde{X} \in T_{e_1 e_2} G(2, 8) \subset \wedge^2(R^8)$, $\tau_* \tilde{X} = X$ is defined by

$$\tilde{X}A = \bar{e}_1 XA.$$

(1) follows from the definitions of \tilde{J} and $J_{e_1 e_2}$. The proof of (2) is easy.

Any section $f: S^6 \rightarrow G(2, 8)$ of the fibre bundle τ defines an almost complex structure J_f on S^6 , $J_{f(v)}: T_v S^6 \rightarrow T_v S^6$. As [4], we give

Definition The section f is holomorphic if $f_* J_f = \tilde{J} f_*$.

From Proposition 3.3 and $\tau f = id$, we have

Proposition 3.4 $J_{f(v)} = \tau_* \tilde{J} f_*$.

For any $X \in T_v^6(S^6)$, $f_*(X) = Z_1 + Z_2$, $Z_1 \in T_{f(v)}^H G(2, 8)$, $Z_2 \in T_{f(v)}^V G(2, 8)$, we have $J_{f(v)}(X) = \tau_* \tilde{J}(Z_1)$. On the other hand, the tangent vector X can be left to a horizontal vector $X^* \in T_{f(v)}^H G(2, 8)$, it is easy to see that $X^* = Z_1$. The almost complex structure $J_{f(v)}: T_v S^6 \rightarrow T_v S^6$ is determined by the complex structure \tilde{J} on $T_{f(v)}^H G(2, 8)$ and the isomorphism $\tau_*: T_{f(v)}^H G(2, 8) \rightarrow T_v S^6$.

By almost complex structure J_f , we have $TS^6 \otimes C = T^{(1,0)} S^6 \oplus T^{(0,1)} S^6$, where $T^{(1,0)} S^6 = \{X - \sqrt{-1} J_f X \mid X \in TS^6\}$, $T^{(0,1)} S^6 = \overline{T^{(1,0)} S^6}$. The almost complex structure J_f is integrable if only if

$$[X, Y] \in \Gamma(T^{(1,0)} S^6) \text{ for any } X, Y \in \Gamma(T^{(1,0)} S^6).$$

Similarly, we have decomposition $TG(2, 8) \otimes C = T^{(1,0)} G(2, 8) \oplus T^{(0,1)} G(2, 8)$ with respect to the complex structure \tilde{J} .

Let ∇ be Riemannian connection on the sphere S^6 and

$$\$(S^6) = \bigcup_{v \in S^6} \{\bar{e}_1 XA \mid X \in T_v S^6 \otimes C\}$$

be a vector bundle over S^6 which is isomorphic to the tangent bundle $TS^6 \otimes C$. The connection ∇ can be generalized to the bundle $\$(S^6)$. We have represented the

tangent space of $G(2, 8)$ as subspace in $\Lambda(R^8)$ or $C\ell_8$. The section $f: S^6 \rightarrow G(2, 8)$ can be viewed as a map $f: S^6 \rightarrow C\ell_8$ and

$$f_*X = Xf$$

for all $X \in \Gamma(TS^6)$ or $X \in \Gamma(TS^6 \otimes C)$. Note that $\tau_*(Xf) = X$, this can also be written as

$$(Xf) \cdot A = X(fA) = X(\bar{e}_1 v A) = \bar{e}_1 XA.$$

The following is the main result of this section.

Theorem 3.5 The almost complex structure J_f is integrable if and only if the map f is holomorphic. Then there is no integrable orthogonal complex structure on the sphere S^6 .

Proof First assuming the map f is holomorphic. For any $X, Y \in \Gamma(T^{(1,0)}S^6)$, we have

$$\tilde{J}f_*(X) = f_*J_{f(v)}X = \sqrt{-1}f_*(X), \quad \tilde{J}f_*(Y) = \sqrt{-1}f_*(Y),$$

then $f_*(X), f_*(Y) \in \Gamma(T^{(1,0)}G(2, 8))$. The Grassmann manifold $G(2, 8)$ is Kaehlerian, $f_*[X, Y] = [f_*(X), f_*(Y)] \in \Gamma(T^{(1,0)}G(2, 8))|_{f(S^6)}$. These shows

$$f_*J_f[X, Y] = \tilde{J}f_*[X, Y] = \sqrt{-1}f_*[X, Y].$$

$f: S^6 \rightarrow G(2, 8)$ is an imbedding, we have $J_f[X, Y] = \sqrt{-1}[X, Y]$ and the almost complex structure J_f is integrable.

Secondly, assuming the almost complex structure J_f is integrable. For any $X, Y \in \Gamma(T^{(1,0)}S^6)$, we have $[X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(T^{(1,0)}S^6)$, then

$$(1 + \sqrt{-1}f)\bar{e}_1 \nabla_X Y A = (1 + \sqrt{-1}f)\bar{e}_1 \nabla_Y X A.$$

From $f\bar{e}_1 X A = \sqrt{-1}\bar{e}_1 X A$ we have

$$\nabla_Y(f\bar{e}_1 X A) = pr(Yf \cdot \bar{e}_1 Y A) + f\bar{e}_1 \nabla_Y X A = \sqrt{-1}\bar{e}_1 \nabla_Y X A,$$

where $pr: S^6 \times (C\ell_8^{\text{even}} \otimes C) \rightarrow \(S^6) is a projection defined naturally. Combine

$$pr(Yf \cdot \bar{e}_1 X A) = (\sqrt{-1} - f)\bar{e}_1 \nabla_Y X A$$

with $(1 + \sqrt{-1}f)\bar{e}_1 \nabla_X Y A = (1 + \sqrt{-1}f)\bar{e}_1 \nabla_Y X A$, we have

$$pr(Yf \cdot \bar{e}_1 X A) = pr(Xf \cdot \bar{e}_1 Y A) = (\sqrt{-1} - f)\bar{e}_1 \nabla_Y X A = (\sqrt{-1} - f)\bar{e}_1 \nabla_Y X A.$$

Denote $\beta(X, Y) = pr(Yf \cdot \bar{e}_1 X A)$.

$\beta(X, Y)$ is a symmetric $C^\infty(S^6)$ -bilinear form on $\Gamma(T^{(1,0)}S^6)$. We shall show that $\beta(X, Y) = 0$ is equivalent to the fact that f is holomorphic. To prove $\beta(X, Y) = 0$ we need only to show $\beta(X, X) = 0$.

Let $\varepsilon_i = \frac{1}{2}(e_{2i-1} - \sqrt{-1}e_{2i})$, $i = 1, 2, 3$, $e_{2i} = J_f e_{2i-1}$, $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be local hermitian frame fields for $T^{(1,0)}S^6$, $\varepsilon_i \varepsilon_j = -\varepsilon_j \varepsilon_i$, $\varepsilon_i \bar{\varepsilon}_i \varepsilon_i = -\varepsilon_i$. The spinor $\sigma = \varepsilon_1 \varepsilon_2 \varepsilon_3$ (or $\varepsilon_1 \varepsilon_2 \varepsilon_3 \bar{\varepsilon}_1 \bar{\varepsilon}_2 \bar{\varepsilon}_3$) generates a spinor bundle on S^6 locally. In the following we prove

$$\varepsilon_j \nabla_{\varepsilon_j} \sigma = 0, \quad j = 1, 2, 3.$$

The proof is due to [4], p.335-341, we write here for complete.

Let $\omega_c = (\sqrt{-1})^3 e_1 e_2 \cdots e_6$ be complex volume element on TS^6 , $\omega_c \cdot \sigma = -\sigma$. For any i ,

$$\varepsilon_i \varepsilon_j \nabla_{\varepsilon_j} \sigma = -\varepsilon_j \varepsilon_i \nabla_{\varepsilon_j} \sigma = -\varepsilon_j \varepsilon_j \nabla_{\varepsilon_i} \sigma = 0,$$

then there is a local function λ_{ε_j} such that

$$\varepsilon_j \nabla_{\varepsilon_j} \sigma = \lambda_{\varepsilon_j} \sigma.$$

On the other hand, $\nabla_{\varepsilon_j}(\omega_c \sigma) = \omega_c \nabla_{\varepsilon_j} \sigma$,

$$\omega_c \cdot \varepsilon_j \nabla_{\varepsilon_j} \sigma = (\omega_c \varepsilon_j \omega_c) \nabla_{\varepsilon_j}(\omega_c \sigma) = \varepsilon_j \nabla_{\varepsilon_j} \sigma.$$

These shows $\varepsilon_j \nabla_{\varepsilon_j} \sigma = 0$ for any j .

From

$$0 = \nabla_{\varepsilon_j}(\varepsilon_j \sigma) = (\nabla_{\varepsilon_j} \varepsilon_j) \sigma + \varepsilon_j \nabla_{\varepsilon_j} \sigma = (\nabla_{\varepsilon_j} \varepsilon_j) \sigma$$

we know that $\nabla_{\varepsilon_j} \varepsilon_j$ is a local $(1,0)$ -frame field. These shows $\beta(\varepsilon_j, \varepsilon_j) = 0$, hence $\beta(X, Y) = pr(Yf \cdot \bar{e}_1 XA) = 0$ for any $X, Y \in \Gamma(T^{(1,0)}S^6)$.

Let $X = (1 - \sqrt{-1}J_f)X_1$, $X_1 \in \Gamma(TS^6 \otimes C)$, $2Yf = (1 - \sqrt{-1}f)Yf + (1 + \sqrt{-1}f)Yf$. By $Yf \cdot f = -f \cdot Yf$, we have

$$(1 - \sqrt{-1}f)Yf(1 - \sqrt{-1}f)\bar{e}_1 X_1 A = (1 - \sqrt{-1}f)(1 + \sqrt{-1}f)Yf\bar{e}_1 X_1 A = 0.$$

Then

$$2pr(Yf \cdot \bar{e}_1 XA) = pr(1 + \sqrt{-1}f)^2 Yf \bar{e}_1 X_1 A = 2pr(1 + \sqrt{-1}f)Yf \bar{e}_1 X_1 A = 0.$$

Let $f_*Y = Yf = Z_1 + Z_2$ where $Z_1 \in \Gamma(T^{(1,0)}G(2, 8))$ and $Z_2 \in \Gamma(T^{(0,1)}G(2, 8))$. By Proposition 3.3, Z_2 is vertical and we can set $Z_2 = (1 + \sqrt{-1}f)(X_2 J_v e_1 + e_1 J_v X_2)$ where $X_2 \in \Gamma(TS^6 \otimes C)$, $f(v) = e_1 J_v e_1$. Then for any $X_1 \in \Gamma(TS^6 \otimes C)$,

$$pr(1 + \sqrt{-1}f)Yf \bar{e}_1 X_1 A = pr(1 + \sqrt{-1}f)^2 (X_2 J_v e_1 + e_1 J_v X_2) \bar{e}_1 X_1 A = 0.$$

Hence $pr(Yf \cdot \bar{e}_1 X A) = 0$ is equivalent to

$$pr(X_2 J_v e_1 + e_1 J_v X_2) \bar{e}_1 X_1 A = 0.$$

Since $X_1 \in \Gamma(TS^6 \otimes C)$ is arbitrarily, $X_2 = 0$, then $f_* Y$ is a $(1, 0)$ -form. We have proved that if the almost complex structure J_f is integrable,

$$\tilde{J}f_*(Y) = \sqrt{-1}f_*(Y) = f_* J_f Y$$

holds for any $Y \in \Gamma(T^{(1,0)}S^6)$. Then f is holomorphic.

If there is an integrable complex structure on the sphere S^6 we have a holomorphic section $f : S^6 \rightarrow G(2, 8)$. Then S^6 is also a Kaehler manifold, this contradict to the fact $H^2(S^6) = 0$.

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E-mail: jwzhou@suda.edu.cn